

Mandelbrot Sets

For historical and aesthetic reasons, we have focused our attention to a parameterized class of transformations $f : \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = z^2 + \lambda$. We will continue to do that in this section, thinking of λ as our parameter. However, it is interesting to note that the game that we are playing with the escape-time algorithm, Julia sets, and various λ s, can be played with all sorts of different functions f . You are encouraged to experiment on the computer with generalizations as simple as $f(z) = z^3 + \lambda$ or as crazy as $f(z) = \lambda \sin(z) - 1$.

Mandelbrot's original question, on which he performed some of the first computer experiments, was the question: "For which values of λ is the Julia set connected?" This changes our way of thinking about Julia sets. The escape-time algorithm was able to plot the filled Julia set for a specific value of λ . This question is asking us to consider *every* $\lambda \in \mathbb{C}$, picture its Julia set, and determine if it is connected. The result is a subset $\mathcal{M} \subset \mathbb{C}$ of λ s that share a common property: namely, that their Julia sets are connected.

It turns out that there is a way to understand the Mandelbrot set without needing to perform a topological analysis of the Julia set for each value of λ . What we are giving here as a definition is given elsewhere as a theorem, sometimes known as the "fundamental theorem of the Mandelbrot set" [Fal06, p. 225].

DEFINITION 6.1. The Mandelbrot set is the set of all $\lambda \in \mathbb{C}$ such that 0 is in the filled Julia set of $f(z) = z^2 + \lambda$. That is,

$$\mathcal{M} = \{\lambda \in \mathbb{C} \text{ such that the orbit of } 0 \text{ under } f(z) = z^2 + \lambda \text{ is bounded}\}$$

In order to begin understanding this definition you can try some values of λ to determine whether or not they are in \mathcal{M} as in the next exercise. Use a calculator, earlier problems, and/or what you know about escape regions to make the computations simpler.

EXERCISE 6.2. Determine whether $\lambda \in \mathcal{M}$ for $\lambda = 0, 1, i, -2, 1 + i$, and .25.

This next example provides a general sense of where the Mandelbrot set lies in the complex plane.

EXERCISE 6.3. Prove that if $|\lambda| > 2$, then $\mathcal{O}(0)$ is unbounded by filling in the blanks on this proof.

PROOF. Let $\lambda \in \mathbb{C}$ with $|\lambda| > 2$. We know from the proof and discussion on page _____ that an escape region for $f(z) = z^2 + \lambda$ takes the form $V = \text{_____}$, where $K = \text{_____}$. So we need only show that there is some n for which $|f^n(0)| > \text{_____}$.

We compute $f(0) = \text{_____}$ and $f^2(0) = \text{_____}$, and so $|f^2(0)| = \text{_____}$, which simplifies to _____. We can apply the triangle inequality result of exercise _____ to see that therefore $|f^2(0)| \geq \text{_____}$. Since $|\lambda| > 2$, this means that _____ $>$ _____ and therefore $|f^2(0)| > \text{_____}$, proving that it is in the escape region and therefore the orbit of 0 is unbounded. \square

Let us consider a sample of λ s of the form $a + bi$, where a ranges from -0.5 to 0.5 and b ranges from 0 to 1 . Figure 1 shows the filled Julia sets for λ s in this range, sampled at increments of $.25$ in both a and b .

Figure 1 gives us a sense of which values of λ may fall into \mathcal{M} . The origin, at center bottom, is definitely in. Looking from there to the right, we see (and have already proved) that $.25$ is in \mathcal{M} and any real $\lambda > .25$ is not. We have shown that $\lambda = i$ is not in \mathcal{M} ; its Julia set appears in the center top and looks like it might be connected, but our calculations in exercise 6.2 have proved it is not.

6.1. Using the escape-time algorithm to plot the Mandelbrot set.

We know from exercise 6.3 any λ in the Mandelbrot set lies in the circle of radius 2 around the origin. Figure 2 shows a Mathematica-generated depiction of the Mandelbrot set generated using an adaptation of the escape-time algorithm. The viewing area is $[-2.1, 2.1] \times [-2.1, 2.1]$, the Mandelbrot set is colored black, and the coloring of the exterior represents the number of iterations before the origin enters the escape region.

Many of the elements of the escape-time algorithm for plotting Julia sets are reused to plot the Mandelbrot set. Here are the ingredients:

- We can use the escape region $V = \{z \in \mathbb{C} \text{ such that } |z| > 2\}$ since the only λ under consideration satisfy $|\lambda| \leq 2$.
- A viewing window of the form $[a, b] \times [c, d]$, i.e. $x + iy \in \mathbb{C}$ with $a \leq x \leq b$ and $c \leq y \leq d$. This time the viewing window is for values of λ .
- We use *numits* and *numgrid* the same way as before, sampling our λ s in a *numgrid* by *numgrid* array. The key difference now is that for each λ , we only need to test whether the orbit of 0 enters the escape region within *numits* iterations.
- A way to have the computer turn this *numgrid* \times *numgrid* array of *js* into colors.

So our Mandelbrot escape-time algorithm is going to cycle through a grid of λ s of our choosing. For each λ , we compute the first *numits* elements of the orbit of 0 and record when, if ever, the origin enters the escape region. If it hasn't by *numits* iterations, we proclaim λ to be in the Mandelbrot set. If not, we assign a color to λ in terms of how long the origin took to enter the escape region. As before, we will have no false negatives but may see false positives. Upping *numits* helps

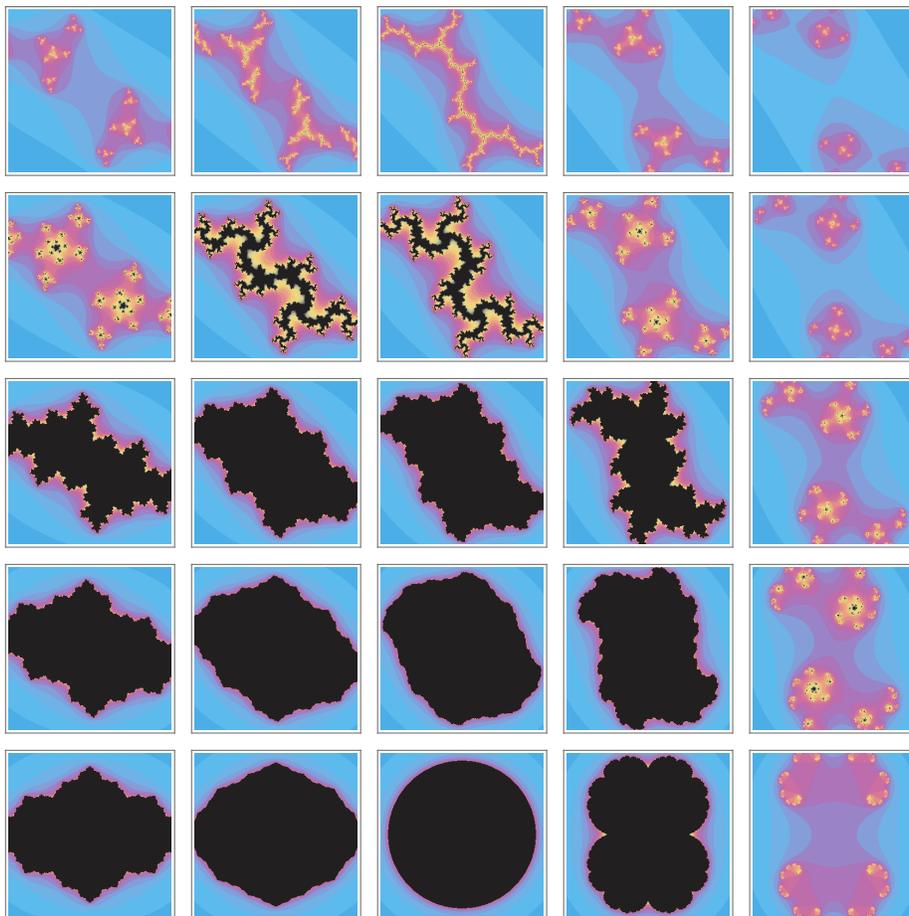


FIGURE 1. Filled Julia sets with $\lambda = -.5$ in the lower left and $\lambda = .5 + i$ in the upper right.

to eliminate the false positives and upping *numgrid* improves the resolution of the plot. We will experiment further with our Mathematica code using this algorithm.

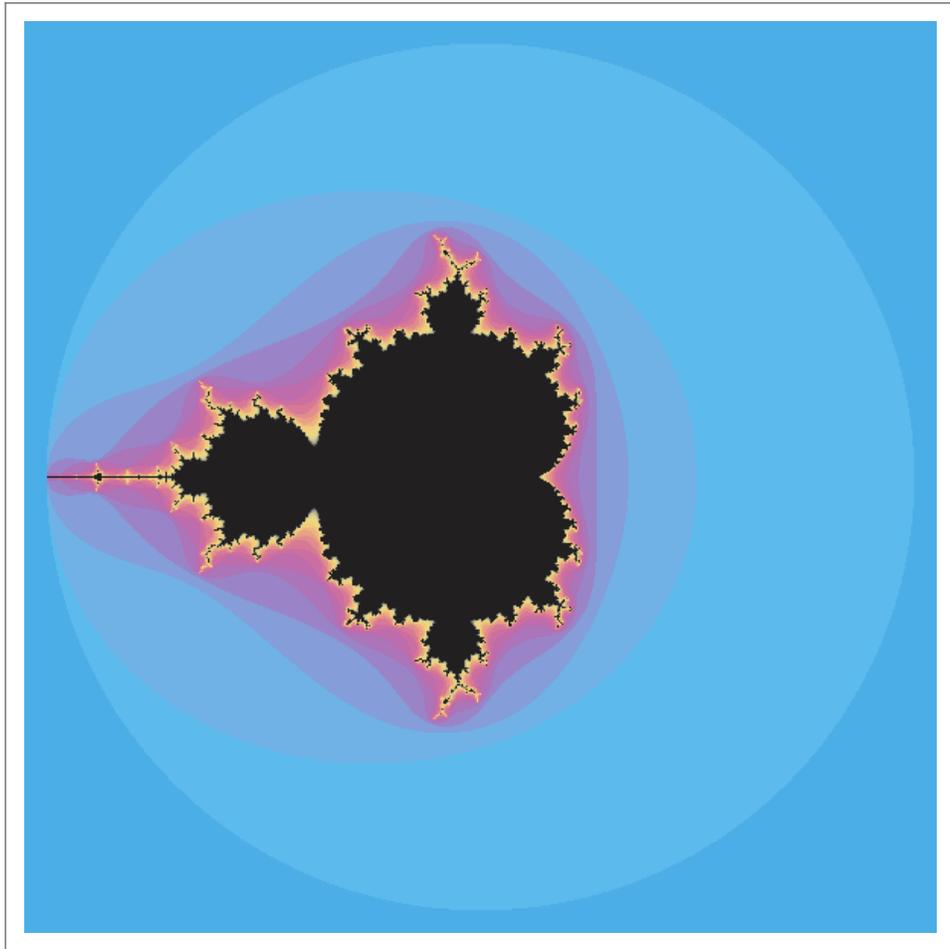


FIGURE 2. A rendering of the Mandelbrot set.