CHAPTER 5

Julia Sets

Our primary source for fractals up to this point has been iterated function systems. We have seen that a collection of contraction maps $T_i : X \to X$, acting together as a collage, have a unique fixed set A. That set A is a collection of points in our original space X that have some or all of the properties identified in section 1.6: a complicated, "fine" structure that is too detailed to be described in traditional geometric terms; that structure is often a form of self-similarity; and the dimension is often not an integer.

Like the attractors A we've been studying, Julia sets are subsets of X that behave in a certain way relative to a transformation. The transformations will no longer be affine,they aren't contractions, and the behavior we study is a little different. We can already find interesting examples when we think about polynomial transformations in the complex plane in section 5.2.

We know from experimentation on iterated function systems that seemingly minor alterations to the T_i's can have unpredictable effects on the set A. Similarly, by changing a single parameter in the transformation we can change its Julia set dramatically. To pique your interest, figure 1 shows two Julia sets that come from the transformation $f(z) = z^2 + \lambda$.

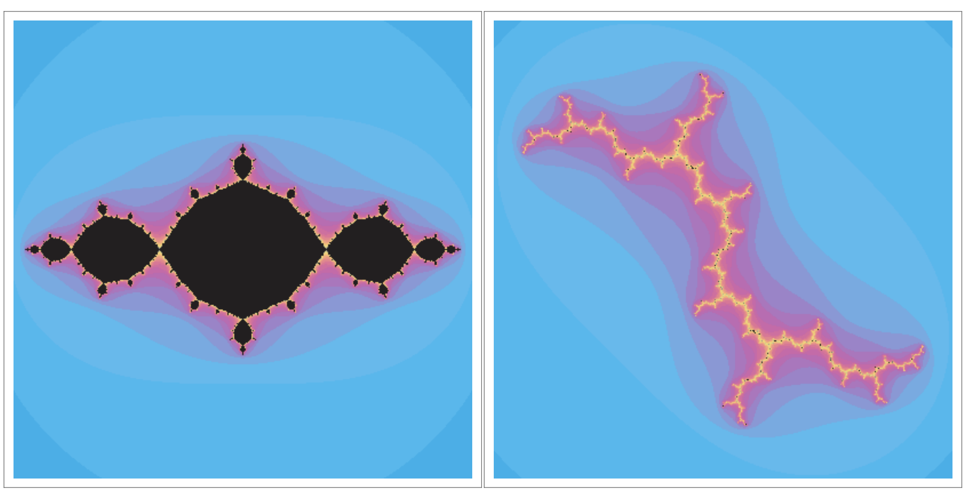

FIGURE 1. The filled Julia sets for $\lambda = 1$ (left) and $\lambda = i$ (right).

5.1. Basic example: dynamical systems in $\mathbb{R}$

We recall some of the notation and terminology from chapter 1 and, with $X \subset \mathbb{R}$, consider some transformation $f : X \to X$. We define $f^2(x)$ to be $f \circ f(x) = f(f(x))$, not $(f(x))^2$, and in general $f^n(x)$ is defined to be $f \circ f \cdots \circ f(x)$ (n times). For $x \in X$ consider the sequence

$$\mathcal{O}(x) = \{x, f(x), f^2(x), f^3(x), ...\}$$

We've called $\mathcal{O}(x)$ the *orbit* of x and we use the notation (X, f) to denote the *dynamical system* defined by f acting on X.

EXERCISE 5.1. Let $A = \mathbb{R}$ and let $f(x) = x^2$. Compute the orbits under f of $x = 2, x = -1$, and $x = 1/2$. Make some sort of graphical depiction of these orbits. Try to give a full qualitative[a] description of the possible behavior of points in this system.

[a] "Qualitative" is used here in contrast to "quantitative". In a qualitative description you give a general description of the behavior that depend on qualities of the initial points. A relevant quality could be something like "falls into this set" or "is at least that big" or some such.

You almost certainly noticed that there are two fixed points of the system in the previous example. One of them counts as attracting and the other repelling. The behavior of orbits near fixed points plays an important role in dynamical systems theory. Another thing you will have noticed is that some of the orbits stay bounded and others go off to infinity.

DEFINITION 5.2. Let $X \subset \mathbb{R}, \mathbb{R}^d$, or $\mathbb{C}$ and let $f : X \to X$ be a polynomial transformation. The *filled Julia set* F_f of f is the set of all points in X whose orbits under f stay bounded. The *Julia set* J_f of f is the boundary[a] of F_f.

[a] The boundary of a set A is the set of all points $x \in X$ such that $B(x, \epsilon)$ contains points in A and not in A for all $\epsilon > 0$. In straightforward examples, the boundary of a set is what you'd expect it to be.

Note that there are more technical characterizations of Julia sets that hold for larger classes of functions. A definition for rational functions, which are quotients of polynomials, is found on [**Bar12**, p. 278], and a definition of 'analytic' functions (ones with convergent Taylor series) appears in [**Fal06**, p. 219] and [**FN**, p. 334]. For the case of polynomial transformations these technical definitions coincide with ours.

EXERCISE 5.3. Identify F_f and J_f for exercise 5.1.

In general, it turns out to be relatively easy to conceptualize F_f and to write computer code to generate images of it. It is more difficult to work with the Julia set definition, which requires us to have an analysis-level comprehension of the definition of boundary points footnoted below. For that reason, in this class we will restrict our attention to filled Julia sets.

EXERCISE 5.4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 2$.

(1) Pick several choices for x and investigate their orbits. Try to find examples of different behavior.
(2) From your experimental points, determine the filled Julia set F_f.

Of course your answer to part (2) isn't a proof, and it would require some more work to make a proof. One way to do it would be to show that if $|x| \geq$ some number, then $|f^n(x)| \to \infty$. This is where the idea of the "escape region" comes in. You can think of an escape region as being all points that are at least K in magnitude and for which $|f(x)|$ is at least as big as $M|x|$, for some $M > 1$. Formally,

DEFINITION 5.5. Let $X \subset \mathbb{R}, \mathbb{R}^d$, or $\mathbb{C}$ and let $f : X \to X$. A region of the form $V = \{x \in X$ such that $|x| > K\}$ is an *escape region* for f if there is an $M > 1$ for which $|f(x)| \geq M|x|$ for all $x \in V$.

EXAMPLE 5.6. Let's try and find an escape region using, say, $M = 2$ for exercise 5.4. We need to find a K that works for the definition. So we need to solve $|f(x)| \geq 2|x|$. Noticing the symmetry that $f(x) = f(-x)$, we work with positive values of x and try to solve $x^2 - 2 \geq 2x$. A little bit of the quadratic formula later we see that if $x \geq 1 + \sqrt{3}$, then $|f(x)| \geq 2|x|$. So an escape region for $f(x) = x^2 - 2$ is the set $V = \{x \in \mathbb{R}$ such that $|x| > 1 + \sqrt{3}\}$.

You may feel like we worked a little bit backwards in that example, but choosing K first and trying to find M was a bit difficult algebraically. For our purposes the precise value of M doesn't matter as much as figuring out what the K is for *some* value of M. Any $M > 1$ will be good enough for the escape-time algorithm, as we shall see soon. For now, though, let us notice a handy way to use escape regions to rule out a point being in F_f.

EXERCISE 5.7. Let $f : X \to X$ and suppose that you have been given an escape region V along with the M and K from the definition.

(1) Show that if $x \in V$, then $f(x) \in V$ also.
(2) Find a lower bound on $|f^n(x)|$ for $x \in V$.
(3) Deduce that if $x \in V$ then $|f^n(x)| \to \infty$ as $n \to \infty$.

This exercise showed that if the orbit of a point ever enters an escape region, then it is destined to wander off to infinity and thus cannot be in the filled Julia set.

EXERCISE 5.8. Investigate the dynamics and compute the filled Julia sets for the following examples.

(1) $f(x) = x + 7$
(2) $f(x) = x/2$
(3) $f(x) = 2x$
(4) $f(x) = x^2/7$

5.2. Classic example: dynamical systems in $\mathbb{C}$.

5.2.1. Review of complex numbers. The complex plane is visualized in much the same way as $\mathbb{R}^2$, with the point $(x, y) \in \mathbb{R}^2$ corresponding to $z = x + iy \in \mathbb{C}$. The horizontal axis consists of the purely real numbers $z = x + 0i \in \mathbb{R}$ and the vertical axis consists of the purely imaginary numbers $z = 0 + iy$. The addition of complex numbers algebraically is $(a+ib)+(c+id) = (a+c)+i(b+d)$. Geometrically this follows the usual parallelogram rule for adding vectors (ak, b) and (c, d) in $\mathbb{R}^2$.

The operation that sets $\mathbb{C}$ apart from $\mathbb{R}^2$ is multiplication. It is defined using the ordinary distributive law

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

and is therefore quite natural to compute.

> EXERCISE 5.9. Let $z_1 = 1 + i, z_2 = 1 + \sqrt{3}i$, and $z_3 = -2i$. Compute $z_i z_j$ for all possible combinations of i and j.

The geometric interpretation of the product of two complex numbers is very interesting and it is essential that you keep it in mind throughout this chapter. To understand it, we must think of z as being written in polar coordinates, so $z = a+ib = r(\cos\theta+i\sin\theta)$, where $r = |z|$ and θ is measured in radians counterclockwise from the positive real axis. In the next exercise you may start to see what happens geometrically with the products you did in the previous exercise.

> EXERCISE 5.10. For the complex numbers in example 5.9,
> (1) Compute r_i and θ_i for $i = 1, 2, 3$, and
> (2) Compute the length and angle of $z_i z_j$ for an assortment of i's and j's of your choosing, looking for a relationship to the lengths and angles of z_i and z_j. Graph z_i, z_j, and $z_i z_j$ on the same set of axes.

The pattern that appears in part (2) is universal and allows us to quickly visualize the product of two complex numbers. All we have to do to get the product is multiply the lengths and add the angles. Let's make that precise.

> EXERCISE 5.11. Show that if $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$, then
> $$zw = rs(\cos(\theta + \phi) + i\sin(\theta + \phi))$$

Our work with Julia sets will primarily use polynomial transformations $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $a_0, ..., a_{n-1} \in \mathbb{C}$. Having a geometric understanding for the terms will be quite helpful to you.

> EXERCISE 5.12. Explain geometrically the relationship between z^n and z for any given complex number z.

> EXERCISE 5.13. Let $z = \cos(\pi/6) + i\sin(\pi/6)$. Make a sketch of z^n for $n = 0, 1, 2, 3, ...$

EXERCISE 5.14. Let $z = \cos(3) + i\sin(3)$ (notice that $\theta = 3$ measured in radians is not a rational multiple of π but it is nonetheless a perfectly good angle). Sketch z, z^2, z^3, and think about what happens to z^n as $n \to \infty$.

5.2.2. The classic example in $\mathbb{C}$. The original transformations Mandelbrot performed computer calculations on in the 1970's was $f(z) = z^2 + \lambda$ [**FN**, Ch. 7], and we will follow in his footsteps. The question he asked was, for which values of λ is the Julia set connected? This is a topological question beyond the scope of our course, but it turns out that the answer can be understood in much more simple language that we will investigate this more in Chapter 6. For now, let's experiment with a few λs.

EXERCISE 5.15. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^2$.

(1) Compute and make a sketch of the orbits of $z = 0, z = \cos(\pi/6) + i\sin(\pi/6), z = \cos(3) + i\sin(3)$, and $z = 2i$.
(2) What is the filled Julia set for this transformation?
(3) Find an escape region for f, showing your choice of K and M.

EXERCISE 5.16. For $\lambda = 1$,

(1) Compute the first few elements of the orbits of $z = 0, z = i$, and $z = 1$.
(2) Find the fixed points of f, if any. At least those are in the filled Julia set.

EXERCISE 5.17. For $\lambda = -1$,

(1) Compute the first few elements of the orbits of $z = 0, z = i$, and $z = 1$.
(2) Try and find an element of the filled Julia set that isn't a fixed point.

There is a fairly simple way to find an escape region for $f(z) = z^2 + \lambda$ for all values of λ. Let $K = \max\{|\lambda|, 2.1\}$. Let $V = \{z \in \mathbb{C} \text{ such that } |z| > K\}$. Let us prove this is an escape region for f.

PROOF. We are interested in showing that $|z^2 + \lambda| \geq M|z|$ for some $M > 1$, where $z \in V$. You can use the triangle inequality to prove that $|z^2 + \lambda| \geq |z^2| - |\lambda|$. Since $|z| \geq |\lambda|$ this implies that $|z^2 + \lambda| \geq |z^2| - |z|$. By our discussion of the multiplication of complex numbers we know that $|z^2| = |z|^2$, so $|z^2 + \lambda| \geq |z|^2 - |z| = |z|(|z| - 1)$. Now we can use the fact that $z \in V$ by noticing that $|z| > K$ implies that $|z| - 1 \geq 2.1 - 1 = 1.1$. This means that $|f(z)| = |z^2 + \lambda| \geq 1.1|z|$ for all $z \in V$ and the definition of an escape region is satisfied. □

Notice that any number of the form $2 + \epsilon$ with $\epsilon > 0$ would have worked in the previous proof. In that case $M = 1 + \epsilon$. As it turns out, you can actually use $K = \max\{|\lambda|, 2\}$ as an escape region without disruption to the escape-time algorithm even though it doesn't quite satisfy our definition.

EXERCISE 5.18. Prove that $|z^2 + \lambda| \geq |z^2| - |\lambda|$ using the triangle inequality.

5.3. The Escape Time Algorithm

Here is the algorithm that was used to create the images in Figure 1. The basic idea is that we test values of z by looking at whether $f^n(z)$ is in our escape region V or not. If we let n be large enough and find that $f^n(z) \notin V$, we surmise (possibly incorrectly) that $z \in F_f$. This algorithm will certainly be accurate about finding points that are *not* in F_f, since we know that as soon as an orbit enters V it is not bounded. We will probably see some false positives, though, especially if n is small, but we are just getting an image and can only be accurate up to the size of a pixel no matter what.

Let's refine this idea more, with the goal of arriving at an algorithm we can put in the computer. The basic way the algorithm works is that it cycles through a grid of points, applies f a given number *numits* times to each point in the grid, and determines whether the result falls into the escape region. If it does, that point is certainly not in F_f and is given a color representing the number of iterations it took to get into the escape region. If the point has not entered the escape region by the time *numits* iterations has been done, it is considered to be in F_f and is colored black.

Here are the elements we need in order to run the algorithm to graph F_f for a given $f : \mathbb{C} \to \mathbb{C}$.

- An escape region V, which we compute beforehand using pencil and paper.
- A viewing window of the form $[a, b] \times [c, d]$, i.e. $x + iy \in \mathbb{C}$ with $a \leq x \leq b$ and $c \leq y \leq d$. The viewing window does not need to include large portions of the escape region since we already know F_f isn't in there.
- The number of points from your window you wish to sample. It's convenient to decide on a grid size *numgrid* and sample z at increments of $(b - a)/numgrid$ horizontally and $(d - c)/numgrid$ vertically.
- Computer code that takes each z from your grid and assigns it a number $j \in \{0, 1, ..., numits\}$ that is either the smallest number for which $f^j(z) \in V$, or it is simply $j = numits$ if the orbit of z does not reach the escape region in *numits* tries.
- A way to have the computer turn this *numgrid* $\times$ *numgrid* array of js into colors.

We will see how these elements are coded into Mathematica for the special case of $f(z) = z^2 + \lambda$ using the (slightly cheating) escape region $V = \{z \in \mathbb{C}$ such that $|z| > 2\}$ we computed in the previous section. Note that upping the sizes of *numgrid* and *numits* will slow the computer down, but we can manipulate them to get trustworthy images.

5.4. Exploring some more, algebraically and Mathematica-ally.

We continue our investigation of the filled Julia sets of $f(z) = z^2 + \lambda$. We have seen that when $\lambda = 0$, the filled Julia set is just the unit disc and the Julia set itself is the unit circle. The unit circle has fractal dimension 1. In our experiments on the computer we have seen that moving λ away from zero produces interesting

Julia sets that at least appear to have a larger fractal dimension. Are there any other values of λ for which the Julia set is simple enough to have fractal dimension equal to one? Where should we expect the Julia sets to lie, and for which λ is the Julia set "interesting" in some way?

EXERCISE 5.19. In this problem we start to think about where the filled Julia set is located in $\mathbb{C}$. Restrict your attention to real values of λ.

(1) The filled Julia set is never completely empty because there is always at least one element of $\mathbb{C}$ whose orbit is bounded. Find a formula in terms of λ for such a point.
(2) Find a condition on λ that guarantees that the filled Julia set contains a number that is not in $\mathbb{R}$.

EXERCISE 5.20. Let $\lambda = -2$.

(1) Find an interval of real numbers that lies in the filled Julia set.
(2) For $z = a+ib$, find a condition on a and b that guarantees that $f(z)$ lies in the escape region $V = \{z \in \mathbb{C} \text{ such that } |z| > 2\}$. (Hint: look at the size of the real part of $f(z)$ and see if the imaginary part adds enough length to get $f(z)$ into V.)
(3) Using your previous answer, make a sketch of $\mathbb{C}$ indicating all values of z whose orbits are sure to escape to infinity.
(4) In mathematica, enlarge the viewing window to include $[-2, 2] \times [-2, 2]$, and make the Julia set for as large of *numits* and *numgrid* as you need to satisfy yourself that you are seeing the true Julia set. Explain where your answer to part (3) fits in the mathematica image.

EXERCISE 5.21. Let's investigate the appearance of the Julia sets for λs in the interval $[.2, .3]$.

(1) Using a calculator or other technology (I found the calculator to be fast), compute the first several points in the orbit of 0 for $\lambda = .2, .25$, and $.3$. Make a conjecture about whether the origin is in the filled Julia set.
(2) To lend credence to your findings in the previous part it helps to think about the fixed points of f. For each case, compute them.
(3) Because λ is real, you know that $f(z) \in \mathbb{R}$ whenever $z \in \mathbb{R}$. That means that you can sketch the graph of $y = f(x)$ to try to understand the orbit of 0. Make a careful sketch of the graph for each case, and include on your sketch the line $y = x$, which should intersect at the fixed points. Use these graphs to verify or refute your conjecture that the origin is in F_f.

Exercise 5.22. Let's focus in on the region around $\lambda = .25$ and let mathematica compute some images.[a]

(1) Alter the mathematica code to plot the Julia sets for the the λs in the previous exercise, going up from $\lambda = .2$ in increments of .0125. What do you notice about how the Julia sets change? Experiment with *numits*. At what level of *numits* does your set of images seem to stabilize?
(2) Choose a smaller range of λs around .25 and repeat with a smaller step size. Experiment with *numits*. At what level of *numits* does your set of images seem to stabilize?
(3) Choose a teeny weeny range around .25 and repeat.

[a] The presenter(s) for this one will need to send me their images before class so I can get them onscreen for everyone.